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The Convergence of the SPH Method

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Abstract—In this paper, we prove the convergence of a smooth particle flow to the solution of a regularized version of the Euler equations describing a generic polytropic fluid. This result, combined with a stability property of the Euler equations with respect to suitable regularizations (result of the same authors of the present paper which will be published elsewhere) allows to achieve the proof of the convergence of the so-called Smoothed Particle Hydrodynamics (SPH) method.

Keywords—Gas dynamics, Lagrangian dynamics, Numerical analysis, Particle numerical methods.

1. INTRODUCTION

The subject of this paper is a numerical technique largely used in fluid-dynamics simulations: the so-called Smoothed Particle Hydrodynamics (SPH).

This particle method has been introduced by Lucy [1] to study self-gravitating fluids.

Basically, the idea of the method is to consider the fluid as an ensemble of (smooth) particles. Each particle has a kernel (a symmetric, regular nonnegative function centered on the particle position) which represents its mass distribution, and carries information on the average values of dynamical and thermodynamical quantities, as well as on their gradients. In general, the kernels may depend on the number of particles (locally or globally). In this work that dependence will be avoided.

Each particle moves in the force field generated by the whole particle system, while the associated quantities evolve under their suitably regularized laws (a more detailed description of this method can be found in [2]).

Necessary requirements for a good numerical method are stability and compatibility with "analytical" equations, and in addition, their solutions must converge to the right solutions. At present, these properties are not proved for SPH. Recently Oelschläger [3] and Di Lisio [4] have obtained convergence results for a particular state law ($P \propto \rho^2$), in the free- and self-gravitating cases, respectively.

The aim of this paper is to establish the convergence of this method (as the number of particles goes to infinity) for a generic polytropic fluid (we stress that the topology used in this work is different from the topology considered in [3,4]).

More precisely, let us consider the motion of a fluid described by the equations:

$$\begin{gathered}\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0, \\ \mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} + \rho^\alpha \nabla \rho = 0, \\ \rho(\mathbf{x}, 0) = \rho_0(\mathbf{x}), \qquad \mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}).\end{gathered} \tag{1}$$

This system represents an ideal fluid with the state law $P = (\alpha+2)^{-1}\rho^{\alpha+2}$. This law is physically interesting when $\alpha > -1$.

We find convenient to put system (1) in the following equivalent Lagrangian form:

$$\begin{gathered}\ddot{\Phi}_t(x) = -(\rho^\alpha \nabla \rho)(\Phi_t(x), t), \\ \int \rho(x,t) f(x)\, dx = \int \rho_0(x) f(\Phi_t(x))\, dx, \\ \Phi_0(x) = x, \qquad \dot{\Phi}_0(x) = \mathbf{v}_0(x), \qquad x \in \operatorname{supp} \rho_0,\end{gathered} \tag{1$'$}$$

where $\Phi_t : \Re^d \times \Re^+ \to \Re^d$ and f is any bounded measurable function.

To establish our result, we need first to regularize the pressure field in (1′). Let us consider a kernel $\delta_\varepsilon \in C_b^2$, such that $\delta_\varepsilon \to \delta$ (the Dirac delta measure) as $\varepsilon \to 0$, and rewrite system (1′):

$$\begin{gathered}\ddot{\Phi}_t(x) = -(\mu_t * \delta_\varepsilon)^\alpha \nabla (\mu_t * \delta_\varepsilon)(\Phi_t(x)), \\ \int \mu_t\,(dx) f(x) = \int \mu_0\,(dx) f(\Phi_t(x)), \\ \Phi_0(x) = x, \qquad \dot{\Phi}_0(x) = \mathbf{v}_0(x), \qquad x \in \operatorname{supp} \mu_0,\end{gathered} \tag{2}$$

where μ_t is a probability measure valued function. The symbol $*$ denotes the usual convolution operation on $\Re^d$.

System (2) allows us to consider density fields which are measure valued functions of time. Moreover, such a system admits solutions globally in time (at least for $\alpha \geq 0$) as follows from the arguments of the present paper.

The convergence of the solutions of system (2) to the solution of (1) as $\varepsilon \to 0$ and for fixed smooth initial conditions, will be the subject of a forthcoming paper [5].

We consider the N particles SPH scheme:

$$\begin{gathered}\ddot{x}_i(t) = -\frac{1}{N}\sum_{j=1}^{N}\left(\frac{1}{N}\sum_{h=1}^{N}\delta_\varepsilon(x_i(t) - x_h(t))\right)^\alpha \nabla\delta_\varepsilon(x_i(t) - x_j(t)), \\ x_i(0) = x_{i0}, \qquad \dot{x}_i(0) = \mathbf{v}_0(x_i(0)), \qquad i = 1, \dots, N.\end{gathered} \tag{3}$$

Using the equivalence $x_i(t) = \Phi_t(x_i(0))$, it is easy to convince oneself that the empirical measure

$$\mu_N(t) = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i(t)} \tag{4}$$

is a solution of (2) for the initial conditions $\mathbf{v}_0$ and $\mu_N(0)$ which is obtained by replacing t by 0 in (4). By the regularity of the force field the solutions $\{x_i(t)\}_{i=1,2,\dots,N}$ of (3) exist for any time t.

Therefore, to prove the convergence of the measure (4) to the solution of problem (2), μ_t, (for a fixed initial velocity field) it is enough to prove the following statement. Let us consider a sequence of measures $\{\mu_0^n\}_{n=1,2,\dots}$ such that $\mu_0^n \to \mu_0$ and the corresponding solutions of the problem (2) $\{\mu_t^n\}_{n=1,2,\dots}$. Then $\mu_t^n \to \mu_t$. All the convergences are in weak convergence topology of measures.

In Section 2, we introduce the basic mathematical tool we shall use to establish the above result, namely the so-called Vasershtein distance. In Section 3, the proof of the previous statement will be given and in Section 4, the convergence of the particle scheme (3) will be shown.

2. THE VASERSHTEIN DISTANCE

Let M be a metric space with bounded metric function $d : M \times M \to \Re^+$. Fixed two probability measures, μ_1 and μ_2, on M, let us consider a joint representation of μ_1 and μ_2, that is a measure P on $M \times M$ such that:

$$\int_{M\times M} f(x_i) P(dx_1, dx_2) = \int_M f(x)\mu_i(dx), \qquad i = 1, 2,$$

for all bounded measurable functions f.

Let $C(\mu_1, \mu_2)$ the set of all joint representations of μ_1 and μ_2. Then,

$$R(\mu_1, \mu_2) = \inf_{P\in C(\mu_1,\mu_2)} \int_{M\times M} P(dx, dy)\, d(x, y) \tag{5}$$

defines a distance in the space of probability measures on M. It has been proved [6] that the topology induced by the metric R is equivalent to the weak convergence topology. The metric (5) is called the Vasershtein distance.

A concrete example giving an idea of the meaning of this distance, is the following.

If $\mu_k = 1/N \sum_{i=1}^{N} \delta_{x_i^k}$ for $k = 1, 2$, then

$$R(\mu_1, \mu_2) = \min_{\pi} \frac{1}{N} d\left(x_i^1, x_{\pi(i)}^2\right), \tag{6}$$

where the above minimum is taken over all the permutations π of $1, \dots, N$. The proof of (6) can be found in [7].

In what follows, the metric space M we shall consider will be $\Re^d$ with the usual metric (notice that this metric is not bounded).

3. THE MAIN RESULT

In this section, we shall prove the statement claimed in the introduction. For a fixed ε, we consider the initial condition (μ_0, v_0) and the corresponding solution (μ_t, Φ_t) of problem (2). Consider also the sequence of initial conditions $\{(\mu_0^n, \mathrm{v}_0)\}_{n=1,2,\dots}$ and the solutions $\{(\mu_t^n, \Phi_t^n)\}_{n=1,2,\dots}$. Then we formulate the main result of the present paper.

THEOREM 1. *Fixed* $\mathrm{v}_0 \in C_b^1$ *and* $\varepsilon > 0$. *Suppose that* $\{\mu_0^n\}_{n=1}^{\infty}$, $\{\mu_0\}$ *are Borel probability measures and that:*

1. $0 \le \alpha < +\infty$;
2. $\delta_\varepsilon(x) > 0$, $\delta_\varepsilon \in C_b^2$;
3. *let* $B(r) = \{x : |x| \le r\}$, *then there exists* $r_0 > 0$ *such that*

$$\int_{B(r_0)} \mu_0^n(dx) = 1, \qquad \int_{B(r_0)} \mu_0(dx) = 1;$$

4. $\mu_0^n \to \mu_0$ *in the weak convergence topology.*

Then,

(i) *There exists a unique pair* (μ_t, Φ_t) *which solves problem (2) with the initial density profile given by* μ_0.

(ii) *If* μ_t^n *solves the problem (2) for all* $n > 0$, *then* $\mu_t^n \to \mu_t$ *in the weak convergence topology, uniformly in* $t \in [0, T]$ *for all* $T > 0$.

The idea underlying the proof is simple. We shall estimate the time derivative of the Vasershtein distance between the measures μ_t and μ_t^n in term of the distance itself and apply the Gronwall Lemma. In what follows, all the positive constants c will depend possibly on ε.

PROOF. We first assume (i) and prove (ii).

Fixed n and t, consider the distance $R(t) \stackrel{\text{def}}{=} R(\mu_t, \mu_t^n)$ between the measures μ_t and μ_t^n. By definition,

$$R(t) \leq \int |x-y| P_t(dx,dy) \stackrel{\text{def}}{=} \int |\Phi_t^n(x) - \Phi_t(y)| P_0(dx,dy), \tag{7}$$

where $P_0(dx,dy)$ is an *arbitrary* joint representation of μ_0^n and μ_0. It is easy to see that $P_t(dx,dy)$ is a joint representation of μ_t^n and μ_t. To simplify the notations, we introduce the quantities

$$F_s^n(x) = (\mu_s^n * \delta_\varepsilon)^\alpha \nabla(\mu_s^n * \delta_\varepsilon)(x), \tag{8}$$

$$F_s(x) = (\mu_s * \delta_\varepsilon)^\alpha \nabla(\mu_s * \delta_\varepsilon)(x). \tag{9}$$

So, we can write

$$\begin{aligned} |\Phi_t^n(x) - \Phi_t(y)| &\leq (1 + t\|\nabla \mathrm{v}_0\|_\infty)|x-y| + \int_0^t ds(t-s)\,|F_s^n(\Phi_s^n(x)) - F_s(\Phi_s(y))| \\ &\leq (1 + t\|\nabla \mathrm{v}_0\|_\infty)|x-y| + \int_0^t ds(t-s)\,|F_s^n(\Phi_s^n(x)) - F_s(\Phi_s^n(x))| \\ &\quad + \int_0^t ds(t-s)|F_s(\Phi_s^n(x)) - F_s(\Phi_s(y))|. \end{aligned} \tag{10}$$

Hence,

$$\begin{aligned} \int |\Phi_t^n(x) - \Phi_t(y)| P_0(dx,dy) &\leq (1 + t\|\nabla \mathrm{v}_0\|_\infty) \int |x-y| P_0(dx,dy) \\ &\quad + \int_0^t ds(t-s) \int |F_s^n(\Phi_s^n(x)) - F_s(\Phi_s^n(x))|\,\mu_0^n(dx) \\ &\quad + \int_0^t ds(t-s) \int |F_s(\Phi_s^n(x)) - F_s(\Phi_s(y))| P_0(dx,dy). \end{aligned} \tag{11}$$

Introducing the quantities:

$$Q(t) = \int |\Phi_t^n(x) - \Phi_t(y)| P_0(dx,dy), \tag{12}$$

$$S(t) = \int_0^t ds(t-s) \int |F_s^n(\Phi_s^n(x)) - F_s(\Phi_s^n(x))|\,\mu_0^n(dx), \tag{13}$$

$$U(t) = \int_0^t ds(t-s) \int |F_s(\Phi_s^n(x)) - F_s(\Phi_s(y))| P_0(dx,dy), \tag{13'}$$

inequality (11) can be rewritten as

$$Q(t) \leq (1 + t\|\nabla \mathrm{v}_0\|_\infty) Q(0) + U(t) + S(t). \tag{14}$$

Now we have to estimate $U(t)$ and $S(t)$. The estimations of the two terms follow the same algebra and lead to the same results, so we shall estimate term S only. To do this, we shall distinguish several cases for α.

We start by considering the case $\alpha = 0$, i.e., the linear case.

In this case, term S can be rewritten as

$$S(t) = \int_0^t ds(t-s) \int |\nabla(\mu_s^n * \delta_\varepsilon)(\Phi_s^n(x)) - \nabla(\mu_s * \delta_\varepsilon)(\Phi_s^n(x))|\,\mu_0^n(dx). \tag{15}$$

But

$$\begin{aligned} &|\nabla(\mu_s^n * \delta_\varepsilon)(z) - \nabla(\mu_s * \delta_\varepsilon)(z)| \\ &\quad = \left| \int \nabla\delta_\varepsilon(z - \tilde{z})(\mu_s^n(d\tilde{z}) - \mu_s(d\tilde{z})) \right| = \left| \int (\nabla\delta_\varepsilon(\xi - z) - \nabla\delta_\varepsilon(\eta - z)) \tilde{P}_s(d\xi, d\eta) \right|, \end{aligned}$$

where $\tilde{P}_s$ is an arbitrary joint representation of μ_s^n and μ_s. So

$$|\nabla(\mu_s^n * \delta_\varepsilon)(z) - \nabla(\mu_s * \delta_\varepsilon)(z)| \leq \|\nabla\nabla\delta_\varepsilon\|_\infty \int |\xi - \eta| \tilde{P}_s(d\xi, d\eta), \tag{16}$$

and finally

$$S(t) \leq c \int_0^t ds(t-s) \int |\xi - \eta| \tilde{P}_s(d\xi, d\eta). \tag{17}$$

Minimizing over all the joint representations set $C(\mu_s, \mu_s^n)$:

$$S(t) \leq c \int_0^t ds(t-s) R(s) \leq c \int_0^t ds(t-s) Q(s), \tag{18}$$

(the same estimate holds for $U(t)$) and substituting in (14), by Gronwall Lemma:

$$Q(t) \leq (1 + t\|\nabla \mathrm{v}_0\|_\infty) Q(0) e^{ct^2}. \tag{19}$$

Minimizing $Q(0)$ over all the joint representations set $C(\mu_0, \mu_0^n)$, by definition (12) and inequality (7) follows:

$$R(0) \to 0 \Rightarrow R(t) \to 0, \tag{20}$$

for any fixed time t. Finally, noting that the regularized pressure field is bounded, that is for any probability measure μ

$$\|\nabla(\mu * \delta_\varepsilon)\|_\infty \leq \|\nabla\delta_\varepsilon\|_\infty, \tag{21}$$

by condition 3 the boundeness of the supports of μ_t and μ_t^n follows. Then the usual metric on $\Re^d$ can be replaced by a suitable bounded metric without modifications. So that the theorem is proved when $\alpha = 0$.

Let us consider the nonlinear cases. Consider first the case $\alpha \geq 1$. Then S can be written as

$$S(t) = \int_0^t ds(t-s) \int |(\mu_s^n * \delta_\varepsilon)^\alpha (\mu_s^n * \nabla\delta_\varepsilon)(\Phi_s^n(x)) - (\mu_s * \delta_\varepsilon)^\alpha (\mu_s * \nabla\delta_\varepsilon)(\Phi_s^n(x))| \mu_0^n(dx). \tag{22}$$

The estimate of the quantity in the modulus follows the same algebra as before:

$$\begin{aligned} |(\mu_s^n * \delta_\varepsilon)^\alpha (\mu_s^n * \nabla\delta_\varepsilon)(z) \; - \; & (\mu_s * \delta_\varepsilon)^\alpha (\mu_s * \nabla\delta_\varepsilon)(z)| \\ & \leq |(\mu_s^n * \nabla\delta_\varepsilon)(z) [(\mu_s^n * \delta_\varepsilon)^\alpha(z) - (\mu_s * \delta_\varepsilon)^\alpha(z)]| \\ & \quad + |(\mu_s * \delta_\varepsilon)^\alpha(z) [(\mu_s^n * \nabla\delta_\varepsilon)(z) - (\mu_s^n * \nabla\delta_\varepsilon)(z)]|. \end{aligned} \tag{23}$$

The second term can be estimated as in the linear case:

$$|(\mu_s * \delta_\varepsilon)^\alpha(z) [(\mu_s^n * \nabla\delta_\varepsilon)(z) - (\mu_s * \nabla\delta_\varepsilon)(z)]| \leq \|\delta_\varepsilon\|_\infty^\alpha \|\nabla\nabla\delta_\varepsilon\|_\infty \int |\xi - \eta| \tilde{P}_s(d\xi, d\eta), \tag{24}$$

where $\tilde{P}_s$ is an arbitrary joint representation in the set $C(\mu_s, \mu_s^n)$.

The first term can be handled as follows:

$$\begin{aligned} |(\mu_s^n * \nabla\delta_\varepsilon)(z) \; & [(\mu_s^n * \delta_\varepsilon)^\alpha(z) - (\mu_s * \delta_\varepsilon)^\alpha(z)]| \\ & \leq \alpha \|\nabla\delta_\varepsilon\|_\infty \|\delta_\varepsilon\|_\infty^{\alpha-1} \left| \int \mu_s^n(d\xi) \delta_\varepsilon(z - \xi) - \int \mu_s(d\eta) \delta_\varepsilon(z - \eta) \right| \\ & = \alpha \|\nabla\delta_\varepsilon\|_\infty \|\delta_\varepsilon\|_\infty^{\alpha-1} \left| \int \tilde{P}_s(d\xi, d\eta)(\delta_\varepsilon(z - \xi) - \delta_\varepsilon(z - \eta)) \right| \\ & \leq \alpha \|\nabla\delta_\varepsilon\|_\infty^2 \|\delta_\varepsilon\|_\infty^{\alpha-1} \int \tilde{P}_s(d\xi, d\eta) |\xi - \eta|, \end{aligned} \tag{25}$$

where $\tilde{P}_s \in C(\mu_s, \mu_s^n)$.

Summarizing:

$$|(\mu_s^n * \delta_\varepsilon)^\alpha (\mu_s^n * \nabla\delta_\varepsilon)(z) - (\mu_s * \delta_\varepsilon)^\alpha(\mu_s * \nabla\delta_\varepsilon)(z)| \le c \int \tilde{P}_s(d\xi, d\eta)|\xi - \eta|, \tag{26}$$

hence,

$$S(t) \le c \int_0^t ds(t-s) \int \tilde{P}_s(d\xi, d\eta)|\xi - \eta|, \tag{27}$$

leading again to the estimations (18), (19), and the statement (20).

Remark. $U(t)$ can be estimated by adding and subtracting in (13′) the quantity:

$$(\mu_s * \delta_\varepsilon)^\alpha(\Phi_s^n(x))\nabla(\mu_s * \delta_\varepsilon)(\Phi_s(y)),$$

obtaining

$$U(t) \le \left(\|\delta_\varepsilon\|_\infty^\alpha + \|\delta_\varepsilon\|_\infty^{\alpha-1}\right)\|\nabla\nabla\delta_\varepsilon\|_\infty \int_0^t (t-s)Q(s). \tag{27'}$$

It remains to consider $0 < \alpha < 1$. This case requires a little more care. In fact, in the computation of the Lipschitz constant of $(\mu_s * \delta_\varepsilon)^\alpha$, we are led to estimate the term $\mu_s * \delta_\varepsilon$ from below. To do this, let $r(s)$ be the radius of the minimal ball containing the support of μ_s. By the equation of motion:

$$r(s) \le r(0) + c\left(s^2 + 1\right), \tag{28}$$

therefore:

$$\inf_{x\in B(r(T))} (\mu_s * \delta_\varepsilon)(x) \ge \inf_{|x|\le r(T))} \delta_\varepsilon(x) \int \mu_s(dy) \ge c(T). \tag{29}$$

The same holds replacing μ_s by μ_s^n. By estimate (29), we can proceed as in the previous case with minor modifications.

It remains to prove (i). This can be done by studying the mapping $\nu_t \to \mu_t$ defined by: $\nu \in C([0,T], M)$, M being the space of the Borel probability measures on $\Re^d$, $\nu_0 = \mu_0$,

$$\begin{aligned}\ddot{\Phi}_t^\nu(x) &= -(\nu_t * \delta_\varepsilon)^\alpha\nabla(\nu_t * \delta_\varepsilon)\left(\Phi_t^\nu(x)\right),\\ \int \mu_t(dx)f(x) &= \int \mu_0(dx)f\left(\Phi_t^\nu(x)\right),\\ \Phi_0^\nu(x) = x, \qquad &\dot{\Phi}_0^\nu(x) = \mathrm{v}_0(x), \qquad x \in \operatorname{supp}\mu_0.\end{aligned} \tag{30}$$

By the same technique, we used to show the convergence, we can also show that this mapping has a unique fixed point for all T. ∎

Repeating the same calculus of Theorem 1, we can investigate the more interesting range $-1 < \alpha < 0$. For this, an hypotheses on the shape of δ_ε is useful.

Theorem 2. *Suppose $-1 < \alpha < 0$, the hypothesis of Theorem 1 and, in addition:*

$$|\nabla\delta_\varepsilon(x)| \le c|\delta_\varepsilon(x)|^{-\alpha}. \tag{31}$$

Then (i) and (ii) of Theorem 1 hold.

Notice that condition (31) is satisfied, for example, by the Gaussian function $\exp(-x^2)$.

Proof. It is enough to prove that $r(t)$ is bounded for all times. Indeed by (31) and Jensen inequality the bound

$$(\mu * \delta_\varepsilon)^\alpha \le \left(c\int \mu(dy)|\nabla\delta_\varepsilon(x-y)|\right)^{-1} \tag{32}$$

holds for any probability measure μ. From this the force field of pressure is bounded and (28) holds. ∎

4. THE CONVERGENCE OF THE PARTICLE METHOD

Theorems 1 and 2 can be directly applied to show the convergence of the sequence of empirical measures (4) $\{\mu_N(t)\}_{N=1,2,\dots}$, whose evolution is stated, for any N, by the equations of motion (3), to the solution of the problem (2), as $N \to \infty$, for positive times.

However, we find useful, for practical purposes, to do some remarks about the hypothesis of the theorems.

We start by noticing that if the points x_i are independent identically distributed random variables of probability μ, then the empirical measures

$$\nu_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i} \qquad N = 1, 2, \dots \tag{33}$$

converge a.e. to the measure μ, that is for any measurable bounded function f:

$$\int \nu_N(dx) f(x) \stackrel{N\to\infty}{\to} \int \mu(dx) f(x) \tag{34}$$

for suitable sequences $x_1, x_2, \dots, x_n, \dots$.

To be more precise, let us consider the quantity

$$\left(\int \nu_N(dx) f(x) - \int \mu(dx) f(x)\right)^2 . \tag{35}$$

The evaluation of its expectation value gives:

$$\begin{aligned} &\int \left(\frac{1}{N}\sum_i f(x_i) - \int \mu(dx) f(x)\right)^2 \mu(dx_1)\dots\mu(dx_N) \\ &\quad = \frac{1}{N}\left[\int f^2(x)\mu(dx) - \left(\int f(x)\mu(dx)\right)^2\right]. \end{aligned} \tag{36}$$

By the boundness of f, the quantity in the square brackets is bounded and the expectation value goes to zero as $N \to \infty$. This means that for almost all the sequences of initial conditions obtained in this way (that is assuming that the initial positions $\{x_{i0}\}_{i=1,2,\dots,N}$ are independent identically distributed random variables with probability μ_0) the empirical measures (4), at time zero, converge to the measure μ_0, as $N \to \infty$, almost surely. So, it is very simple to satisfy Hypotheses 3 and 4 of Theorem 1.

It is also easy but much more expensive from a computational point of view, to satisfy the hypothesis on the kernel δ_ε. In fact, the kernels most commonly used in numerical simulations are spline functions with compact support and, to minimize the amount of calculus, this support is, usually, small with respect to the scale of the system. Moreover, condition (31) is satisfied, for fixed α, only if the degree of the spline is large enough.

In conclusion, the convergence of the SPH method is shown when the limiting equations are suitably regularized. In a following paper [5], the convergence of the solution of this regularized problem (problem (2)) to the solution of the problem (1) will be shown for short times. In this way, the convergence of the SPH method will be completely proved.

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